# THE AMPLITUDE DEPENDENT SPIN TUNE AND THE INVARIANT SPIN FIELD IN HIGH ENERGY PROTON ACCELERATORS

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There is some ambiguity in the definitions of spin tune and spin-orbit resonance on synchro-betatron orbits. We clarify these issues and provide a numerical illustration of the internal consistency of our definitions using the computer code SPRINT for HERA with a standard Siberian Snake arrangement.

## 1 Introduction

The motion of the unit spin vector $\hat{S}$ in the rest frame of a relativistic charged particle traveling in electric and magnetic fields is governed by the Thomas-BMT precession equation, which takes the form $d\hat{S}/d\theta = \vec{\Omega}(\gamma, \vec{z}, \theta) \times \hat{S}$ where $\theta$ is the azimuthal position in the ring, $\vec{z}$ is the position in phase space and $\gamma = E/m$ is the Lorentz factor.

In a purely vertical magnetic field, the spin precesses around the vertical direction. But particles on synchro–betatron trajectories experience additional $\theta$ and $\vec{z}$ dependent fields. Then if the spin precession frequency is in resonance with the orbital frequencies the attainable polarization can become very small. We now investigate this in detail for protons at very high energy.

## 2 The Invariant Spin Field

In the following we will assume the equations of orbital motion to be integrable: $\vec{z} = \vec{z}(\vec{J}, \vec{\Psi})$, $\vec{J} =$ const., $\vec{\Psi}(\theta) = \vec{\Psi}_0 + \vec{Q}\,\theta$. An arbitrary initial spin $\hat{S}_i$ is mapped to a final spin $\hat{S}_f$ by means of an orthogonal spin transport map $\underline{R}$

$$\hat{S}_f = \underline{R}_{\vec{J}}(\theta_f, \theta_i; \vec{\Psi}_i)\; \hat{S}_i \quad . \tag{1}$$

We define a general spin field on a torus $\vec{J} =$ const. as a function $\hat{f} : [0, 2\pi)^3 \times \mathbb{R} \mapsto \mathbb{S}_3$ that maps the "angle space" $\times$ azimuthal domain onto the unit sphere and is, if evaluated along each orbital trajectory $\vec{\Psi}_0 + \vec{Q}\,\theta$, a solution of the T–BMT equation

$$\frac{d}{d\theta}\hat{f}_{\vec{J}} \equiv \left[\partial_\theta + \vec{Q} \cdot \partial_{\vec{\Psi}}\right]\; \hat{f}_{\vec{J}} = \vec{\Omega} \times \hat{f}_{\vec{J}} \quad . \tag{2}$$

The invariant spin field ($\equiv \hat{n}$–axis )[1,2,3] is defined to be the special spin field that is periodic w.r.t. $\theta$, $\hat{n}_{\vec{J}}(\vec{\Psi},\theta) = \hat{n}_{\vec{J}}(\vec{\Psi},\theta+2\pi)$ and it describes the equilibrium polarization distribution of the beam. [4] The evolution equation of $\hat{n}$ under $\underline{R}_{\vec{J}}$ reads as

$$\hat{n}_{\vec{J}}(\vec{\Psi}_f,\theta_f) = \underline{R}_{\vec{J}}(\theta_f,\theta_i;\vec{\Psi}_i)\,\hat{n}_{\vec{J}}(\vec{\Psi}_i,\theta_i)\ \ \forall\,\theta_f,\theta_i,\vec{\Psi}_f = \vec{\Psi}_i + (\theta_f-\theta_i)\vec{Q}\ \ . \quad (3)$$

Since the spin action $I \equiv \hat{S}(\theta;\vec{z}_i,\hat{S}_i)\cdot\hat{n}_{\vec{J}}(\vec{\Psi}(\theta),\theta)$ is conserved along a trajectory, the time averaged polarization on some torus cannot be greater than the static polarization limit defined by the directional spread of the $\hat{n}$–axis on that torus

$$P_{\lim}(\vec{J},\theta) \equiv \left\| \left\langle \hat{n}_{\vec{J}}(\vec{\Psi},\theta) \right\rangle_{\vec{\Psi}} \right\|\ \ . \quad (4)$$

Note that the unit eigenvector $\hat{e}_R(\vec{J},\vec{\Psi},\theta) = \underline{R}_{\vec{J}}(\theta+2\pi,\theta;\vec{\Psi})\ \hat{e}_R(\vec{J},\vec{\Psi},\theta)$ of the one turn spin map approaches $\hat{n}_{\vec{J}}$ only in the limit $\vec{J}\to\vec{0}$. Then $\hat{n}_0(\theta) \equiv \hat{n}_{\vec{0}}(\vec{\Psi},\theta) \equiv \hat{e}_R(\vec{0},\vec{\Psi},\theta)$. A spin parallel to $\hat{e}_R(\vec{J},\vec{\Psi},\theta)$ at $\vec{\Psi}$ is parallel to $\hat{e}_R(\vec{J},\vec{\Psi},\theta)$ after one turn but the orbital phase has become $\vec{\Psi}+2\pi\vec{Q}$ and generally $\hat{e}_R(\vec{J},\vec{\Psi},\theta) \neq \hat{e}_R(\vec{J},\vec{\Psi}+2\pi\vec{Q},\theta)$ so that $\hat{S}(\theta;\vec{z}_i,\hat{S}_i)\cdot\hat{e}_R(\vec{J},\vec{\Psi},\theta)$ is *not* conserved.

## 3 The Amplitude Dependent Spin Tune

In order to obtain a complete action–angle representation of spin motion [5] we must assign to each point in phase space an orthonormal coordinate system $(\hat{n}_{\vec{J}},\hat{u}^1_{\vec{J}},\hat{u}^2_{\vec{J}})$ which is periodic in $\vec{\Psi}$ and $\theta$ and in which the spin precession rate around $\hat{n}$ is constant along each trajectory and independent of starting azimuth and orbital phases. The spin phase advance per turn in this frame divided by $2\pi$ is called the spin tune $\nu(\vec{J},\gamma)$. For each arbitrary initial spin $\hat{S}_i$ at $(\vec{\Psi}_i,\theta_i=0)$ we can compute $I = \hat{S}_i\cdot\hat{n}(\vec{\Psi}_i,0)$ and $\Phi_0 = \arctan(\hat{S}_i\cdot\hat{u}^2(\vec{\Psi}_i,0)/\hat{S}_i\cdot\hat{u}^1(\vec{\Psi}_i,0))$. Then the spin motion can be written in the form

$$\hat{S}(\theta) = \sqrt{1-I^2}\,\Re\left\{(\hat{u}^1+i\hat{u}^2)\,e^{-i(\nu\theta+\Phi_0)}\right\} + I\,\hat{n} \quad (5)$$

which simply describes a rotation around $\hat{n}$ with constant rate. To compute the spin tune on a trajectory, we begin with *some* unit vectors $\tilde{u}^1,\tilde{u}^2$, periodic in $\vec{\Psi}$ and $\theta$ and in the plane perpendicular to $\hat{n}$. The spin precession frequency $\tilde{\nu}(\vec{J},\gamma;\vec{\Psi},\theta)$ in the $\tilde{u}^1,\tilde{u}^2$–system is periodic in $\vec{\Psi}$ and $\theta$. One easily shows [2,3] that $\nu(\vec{J},\gamma)$ is given by the average of $\tilde{\nu}$ along a trajectory $\vec{\Psi}(\theta)$

$$\nu(\vec{J},\gamma) = \lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta \tilde{\nu}(\vec{J},\gamma;\vec{\Psi}(\theta'),\theta')\,d\theta'\ \ . \quad (6)$$

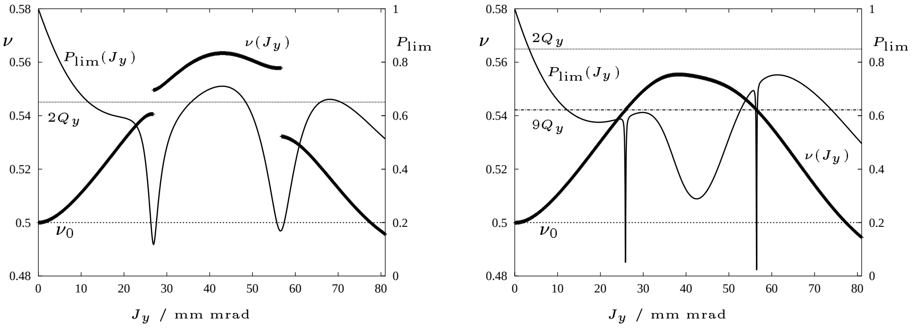


Figure 1: The amplitude dependent spin tune $\nu$ and the static polarization limit $P_{\rm lim}$ vs. vertical orbital action $J_y$ as calculated with `SPRINT` for the HERA–$p$. Left: vertical tune $Q_y = 32.2725$, right: $Q_y = 32.2825$.

Having obtained the spin tune we next construct $\hat{u}^1(\vec{\Psi}(\theta),\theta)$ by starting with $\hat{u}^1(\vec{\Psi}(0),0) = \tilde{u}^1(\vec{\Psi}(0),0)$ and subsequently rotating the vector $\tilde{u}^1(\vec{\Psi}(\theta),\theta)$ around $\hat{n}$ by the angle $\int_0^\theta \tilde{\nu}(\vec{\Psi}(\theta'),\theta')d\theta' - \nu\theta$. Then finally $\hat{u}^2 = \hat{n} \times \hat{u}^1$ so that the complete periodic $(\hat{n},\hat{u}^1,\hat{u}^2)$–system has been found. The spin tune on the closed orbit $\nu_0(\gamma)$ is just $\nu(\vec{0},\gamma)$. In the purely vertical field on the closed orbit of a perfectly flat ring $\nu_0 = G\gamma$ where $G = g/2 - 1 \approx 1.7928$ is the gyromagnetic anomaly of the proton.

The vectors $\hat{u}^1, \hat{u}^2$ as well as the amplitude dependent spin tune are not unique. We can use another periodic system $\hat{u}^1_k, \hat{u}^2_k$ which differs from $\hat{u}^1, \hat{u}^2$ by some uniform rotation of $2\pi(k_0 + \vec{k}\cdot\vec{Q})$ per turn around $\hat{n}$. The corresponding spin tune will be $\nu + k_0 + \vec{k}\cdot\vec{Q}$. But there is normally only one branch for which $\lim_{\vec{J}\to\vec{0}} \nu(\vec{J},\gamma) = \nu_0(\gamma)$. If the spin tune is in resonance with the orbital tunes: $\nu(\vec{J},\gamma) = m_0 + \vec{m}\cdot\vec{Q}$, we can find a coordinate system in which an arbitrary spin does not precess. Then the $\hat{n}$–axis is not unique. Moreover near to strong resonances $\hat{n}$ is a strongly varying function of $\vec{\Psi}$ and $P_{\rm lim}$ can be small.

Fig. 1 shows the amplitude dependent spin tune $\nu$ and the static polarization limit $P_{\rm lim}$ plotted vs. vertical orbital action $J_y$ as calculated with `SPRINT` for the HERA–$p$ '96–luminosity optics with 4 Siberian Snakes (1 long. and 3 rad.) and flattening snakes around the East- North- and South–IP. The reference momentum is 805 GeV and the radial and longitudinal orbital degrees of freedom are not excited ($J_x = J_z = 0$). Fig. 1 (left) corresponds to a fractional vertical tune of $[Q_y] = .2725$, whereas fig. 1 (right) has $[Q_y] = .2825$. Starting at $J_y = 0$ on the design orbit with $P_{\rm lim} = 1$ and $\nu(0,\gamma) = \nu_0 = \frac{1}{2}$, $P_{\rm lim}$ smoothly decreases and $\nu$ increases with increasing $J_y \lesssim 20$ mm mrad. In fig. 1 (left) the spin tune levels off near the resonance condition $\nu = 2Q_y$ before it makes a symmetrical jump across the resonance line exactly at the orbital amplitude ($\approx$ 27 mm mrad) where $P_{\text{lim}}$ has a pronounced minimum. Then after reversing its slope $\nu$ jumps over the resonance line in the other direction at $J_y \approx 56$ mm mrad where $P_{\text{lim}}$ has a second sharp minimum. Finally, $P_{\text{lim}}$ decays asymptotically as expected. In fig. 1 (right) $Q_y$ is too large to allow the line $\nu = 2Q_y$ to be crossed but as $\nu$ gets close to $2Q_y$ the static polarization limit has a rather wide but moderate minimum. With this $Q_y$ the 9-th order (!) resonance $\nu = -2+9Q_y \approx \frac{1}{2}$ is crossed at $J_y \approx 26$ and 56 mm mrad where $P_{\text{lim}}$ has two sharp minima. Thus, as expected, $P_{\text{lim}}$ is small in the neighborhood of spin orbit resonances. Note that owing to the step size in our simulations we see no symmetric spin tune jumps in fig. 1 (right) but preliminary indications are that the spin orbit resonance condition is never *exactly* fulfilled and that the size of the jumps decreases with increasing resonance order.

The spin tune $\nu$ as in eqs. (5,6) *cannot* be obtained from the phase $\mu_R$ of the two complex eigenvalues of the one turn map $\underline{R}(\theta + 2\pi, \theta; \vec{\Psi})$. That fake spin tune would not describe a rotation at constant rate around $\hat{n}_{\vec{J}}(\vec{\Psi}(\theta), \theta)$. It represents an rotation in a *fixed* coordinate system defined by $\hat{e}_R(\vec{J}, \vec{\Psi}_i, \theta_i)$ and two vectors in the plane perpendicular to $\hat{e}_R$ at $\vec{\Psi}_i$. [4] We have analysed the phase average of the fake spin tune also and not observed any correlation between the behavior of $P_{\text{lim}}$ and $\langle \mu_R \rangle_{\Psi_y}$. [4]

The proper definition of the amplitude dependent spin tune $\nu(\vec{J}, \gamma)$ involves the invariant spin field $\hat{n}_{\vec{J}}$ and spin action-angle variables. $\nu(\vec{J}, \gamma)$ can be computed by means of stroboscopic averaging [3,4] and also by Fourier analysis of the one turn spin map [6]. Both methods are implemented in `SPRINT` and give the same result. The concepts of the invariant spin field and the amplitude dependent spin tune allow depolarizing resonances to be identified and classified in a rigorous way.

## References


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